Passive Network Synthesis without Transformers

In honour of Yutaka Yamamoto’s 60th birthday

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with

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The three linear, passive two-terminal electrical elements

resistor capacitor inductor

Why three?
SYNTHESIS OF A FINITE TWO-TERMINAL NETWORK WHOSE DRIVING-POINT IMPEDANCE IS A PRESCRIBED FUNCTION OF FREQUENCY

By Otto Brune

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PART I. INTRODUCTION

1. Statement of the Problem

In the well known methods of analysing the performance of linear passive electrical networks with lumped network elements it is usual to derive from the given structure of the network a scalar function $Z(\lambda)$ known as the impedance function of the network; this function determines completely the performance

$^1$Containing the principal results of a research submitted for a doctor’s degree in the Department of Electrical Engineering, Massachusetts Institute of Technology. The author is indebted to Dr. W. Cauer who suggested this research.

O. Brune (1931): the driving-point impedance of a linear passive two-terminal network is positive-real. Conversely:

Any (rational) positive-real function can be realised using resistors, capacitors, inductors and transformers.
R. Bott and R.J. Duffin showed that transformers were unnecessary in the synthesis of positive-real functions. (1949)
Foster preamble for a positive-real $Z(s)$

Removal of poles on $j\mathbb{R} \cup \{\infty\}$

$$Z = sL + Z_1, \quad (Z_1 \text{ proper})$$

Removal of zeros on $j\mathbb{R} \cup \{\infty\}$

$$Z = \left( \frac{As}{s^2 + \omega^2} + Y_1 \right)^{-1}$$

Subtract minimum real part

$$Z = R + Z_2$$

Not necessarily a unique process
Minimum functions

A **minimum function** $Z(s)$ is a positive-real function with no poles or zeros on $j\mathbb{R} \cup \{\infty\}$ and with the real part of $Z(j\omega)$ equal to 0 at one or more frequencies.
Biquadratic minimum function

\[ Z(s) = \frac{As^2 + Bs + C}{Ds^2 + Es + F} \]

with \( A, B, \ldots, F > 0 \) and \( BE = (\sqrt{AF} - \sqrt{CD})^2 > 0 \) is a minimum function. Bott-Duffin realisation:

3 capacitors, 3 inductors and 2 resistors!!
Some of the literature on RLC synthesis


G. Dittmer, Zur Realisierung von RLC-Brückenzweipolen mit zwei Reaktanzen und mehr als drei Widerständen (On the realisation of RLC two-terminal bridge networks with two reactive and more than three resistive elements), Nachrichtentechnische Zeitschrift, 225-230, 1970.

Ladenheim’s master’s thesis (1948)

Ladenheim considered all networks with at most five elements and at most two reactive elements, and reduced the whole set to 108 networks (1948).

Questions not answered:

▶ What is the totality of biquadratics which may be realised?
▶ How many different networks are needed?
The Concept of Regular Positive Real Functions

Definition
A positive-real function $Z(s)$ is defined to be *regular* if the smallest value of $\text{Re} \left( Z(j\omega) \right)$ or $\text{Re} \left( Z^{-1}(j\omega) \right)$ occurs at $\omega = 0$ or $\omega = \infty$.

Example 1. $Z_1(s) = \left( \frac{s+2}{s+1} \right)^2$

Smallest value of $\text{Re} \left( Z_1(j\omega) \right)$ occurs at $\omega = \infty$, hence $Z_1(s)$ is regular.
The Concept of Regular Positive Real Functions

Example 2. $Z_2(s) = \left(\frac{s+5}{s+1}\right)^2$, which is non-regular:

![Graph of Z_2(s)](image)

$Y_2(s) = 1/Z_2(s)$

![Graph of Y_2(s)](image)
Network Quartets

- Exchange of capacitors and inductors
- Duality (series ⇄ parallel, capacitor ⇄ inductor)

Lemma 1.
$N_a$ is regular $\implies N_b, N_c, N_d$ are all regular.

A network quartet which has only one network:
Properties of Regular Positive Real Functions

**Lemma 2**
The following networks are always regular:

**Lemma 3**
A network that has all reactive elements of the same kind can only realise regular immittances.

**Lemma 4**
The following networks are always regular:
Five-Element Series-Parallel Networks with Two Reactive Elements—Elimination Process

Assume structure 11 can be non-regular:

Contradiction.
Five-Element Series-Parallel Networks with Two Reactive Elements

Theorem 1
A biquadratic impedance can be realised by series-parallel five-element networks with two reactive elements if and only if it is regular. Moreover, the following two network quartets cover all cases.

Five-Element Bridge Networks with Two Reactive Elements

Theorem 2
Bridge networks with two reactive and three resistive elements can only realise regular immittances except for the third network quartet.
A canonical form and the regular region

Extraction of a constant multiplier and frequency scaling gives a canonical form for biquadratics:

\[ Z_c(s) = \frac{s^2 + 2U \sqrt{W} s + W}{s^2 + \left( 2V / \sqrt{W} \right) s + 1/W}, \]

where \( U, V, W > 0. \)

Positive-real \( \Rightarrow \ \sigma_c \geq 0 \)

Regular \( \Rightarrow \ \lambda_c, \lambda_c^\dagger \geq 0 \)
The impedances that can be realised by the third bridge network quartet with $W \in (1/3, 1)$.
Theorem
It is possible to build a passive mechanism of small mass whose impedance (velocity/force) is any rational positive-real function.

Proof
Bott-Duffin + ideal inerter: \( F = b(\ddot{x}_1 - \ddot{x}_2) \), where physical embodiments must satisfy:

- Inertance \( b \) (kg) is independent of mass;
- Inertance is independent of travel.
Ballscrew inerter made in Cambridge University Engineering Department (2003)

Mass $\approx 1$ kg, Inertance (adjustable) = 60–180 kg
Happy 60th Birthday
Yutaka

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Yutaka’s Birthday Puzzle