Equivalence of Internal and External Stability for a Class of Distributed Systems*

Yutaka Yamamoto†

Abstract. It is well known that for infinite-dimensional systems, exponential stability is not necessarily determined by the location of spectrum. Similarly, transfer functions in the $H^\infty$ space need not possess an exponentially stable realization. This paper addresses this problem for a class of impulse responses called pseudofrational. In this class, it is shown that the difficulty is related to classical complex analysis, especially that of entire functions of exponential type. The infinite-product representation for such entire functions makes it possible to prove that stability is indeed determined by the location of spectrum or by a modified $H^\infty$ condition. Examples are given to illustrate the theory.

Key words. External stability, Distributed systems, Realizations.

1. Introduction

It is a widely appreciated fact that the space $H^\infty(C,)$ plays a key role in a variety of important system/control problems for finite-dimensional systems, e.g., robust stability/stabilizability. This is crucially based on the fact that if a transfer function belongs to $H^\infty$, then its minimal (canonical) realization is stable. This fact is still further supported by the fact that if a finite-dimensional system has its spectrum contained in the open left-half plane, then it is exponentially stable. The same approach is also employed to study a class of distributed parameter systems, for example, [CDJ], [KP], and [CG].

There is, however, a marked difference between the two. While in the finite-dimensional case, $H^\infty$ guarantees the strongest notion of stability, i.e., internal exponential stability, it is not necessarily guaranteed for the infinite-dimensional case, and only a weaker notion of $L^2$ bounded input/output stability is, in general, guaranteed. In fact, Zabczyk [Z] gave an example of a system in a Hilbert space whose state transition grows as rapidly as any prescribed exponential order yet its spectrum stays in an arbitrarily prescribed left-half complex plane (see also [HP] and [PZ] in this regard).

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† Department of Applied Systems Science, Faculty of Engineering, Kyoto University, Kyoto 606, Japan.
An example given by Logemann [L2] is even more striking. The transfer function

\[ W(s) = \frac{1}{s + 1} \frac{1}{(1 - e^{-\tau}) + a^2} \quad \tau > 0, \quad a > 0, \]  

(1)

belongs to \( H^\infty(C_\infty) \), maps \( L^2 \) inputs to \( L^2 \) outputs, yet its canonical (i.e., approximately reachable and topologically observable; see Section 2 below) realization is not exponentially stable.

This discrepancy has attracted recent research interest, and there are now a number of investigations. Jacobson and Nett [JN] and Callier and Winkin [CW] have worked with the algebra \( \mathcal{H} \) of transfer functions which are expressible as a ratio of functions in \( \mathcal{A} \) (see [CD] with the denominator being invertible in a right-half complex plane. They proved the equivalence of internal and external stability in the above sense under the hypotheses of (i) bounded input/output operators, and (ii) the system is both stabilizable and detectable. Since the first assumption is restrictive in dealing with delay or boundary-control systems, Curtain [C] generalized their results to those with unbounded input/output operators. These results, however, do not apply to Logemann's example, since in his example, the system is, although canonical, not stabilizable or detectable due to infinity many poles approaching the imaginary axis. This situation is grossly different from the finite-dimensional case. Logemann [L1] also derived this equivalence of stability for neutral or integrodifferential systems, making use of stability criteria for these systems. However, the class of systems where such equivalence is shown is not specified in terms of input/output relations.

The works above start with a concrete model and then derive conditions on the model under which the \( H^\infty \)-property of transfer functions is equivalent to exponential stability. A different approach was taken by Yamamoto and Hara [YH1]. They start with a specialized class (called pseudorational) \( \mathcal{A} \) of transfer functions, and then derive conditions for canonical realizations to be exponentially stable. The condition there involves higher-order estimates on the imaginary axis (e.g., \( \phi(t) \in H^\infty(C_\infty) \), etc., where \( \phi(t) \) is the denominator of the transfer function; see Section 5 below). Logemann's example is seen to satisfy the zero-order condition (i.e., in \( \mathcal{F}(C_\infty) \)), but not the higher-order condition on the imaginary axis. However, the condition there is difficult to apply to the robust stability problem, and, in particular, the question whether the internal stability is determined by the location of spectrum remains open.

Why such complication in stability questions for infinite-dimensional systems? Many aspects are involved. Let us list some of them:

1. There are, in general, infinitely many poles.
2. Such poles may asymptote a line parallel to the imaginary axis.
3. The least upper bound of the real part of the poles need not determine the growth [Z], [HP].

They are largely of a complex analytic nature. Point 3 above says that stability is not determined by singularities in utmost generality. This leads to the question of what class of impulse responses (transfer functions) and what realizations we should consider. We employ the class of pseudorational impulse responses as considered.
in [VJ], [Y4], [YHJ], etc. This class has the following features:

1. It is large enough to include all delay systems, neutral or retarded, with point or distributed delays.

2. It admits a nice representation for transfer functions, that is, a transfer function is the ratio of two entire (i.e., holomorphic on the whole planes) functions of exponential type.

Making use of these properties, we investigate the properties of transfer functions in this class. The theory reveals a close connection with the classical theory of infinite-products of entire functions of exponential type due to Hadamard, Lindelöf, and others. In particular, we see that in this class of transfer functions, the distribution of poles determines the growth order of the transfer function. This enables us to prove that the canonical realization of a pseudorational impulse response is exponentially stable if the least upper bound of the real part of the zeros of the denominator of the transfer function is negative. Finally, examples are discussed to illustrate the results.

Notation and Convention

For a distribution \( a \) in support \( s \) is the smallest closed set outside of which \( a \) is zero. As usual (see [S2] and [T]), \( \mathcal{D}(\mathbb{R}^n) \) is the space of distributions having compact support \( I, s, 0 \); for example, the Dirac distribution \( \delta \) at the origin, its derivative \( \delta' \), Dirac distribution \( \delta_0 \) on a set \( a \), etc., are elements in \( \mathcal{D}(\mathbb{R}^n) \).

For a distribution \( s \), its order \( \mathcal{O} \) is denoted by \( a \). Roughly speaking, it is the number of essential differentiations in the action of \( s \). For example, \( \delta = \delta \), \( \delta' = \delta_1 \), etc. \( \mathcal{F}(\mathbb{R}) \) is the space of distributions having support bounded on the left. Clearly \( \mathcal{D}(\mathbb{R}^n) \) is a subspace of \( \mathcal{F}(\mathbb{R}) \). Both \( \mathcal{D}(\mathbb{R}^n) \) and \( \mathcal{F}(\mathbb{R}) \) constitute a convolution algebra.

For an element \( x \in \mathcal{F}(\mathbb{R}) \), we often need to restrict it to the half-line \([0, \infty)\). The truncation operator \( \psi \), defined by \( \psi_x := \phi_{x, \infty} \), can be extended to distributions by restricting their actions to those \( C^\infty \)-functions whose supports are contained in \([0, \infty)\). That is,

\[
\langle \psi_x, \phi \rangle := \langle x, \phi \rangle, \quad \supp \phi < [0, \infty).
\]

(2)

The space \( \mathcal{O}^\infty := \bigcup_{n=0} \mathcal{L}_n^2([-\infty, 0])^n \), with the inductive limit topology (see [S1] and [T]), is called the space of inputs, and \( \Gamma^* := \bigcup_{n=0} \mathcal{L}_n^2([0, \infty])^n \), i.e., the p-product of the space of locally Lebesgue square integrable functions on \([0, \infty)\), is called the space of outputs. With respect to the family of \( L^2 \)-norms on all bounded intervals, \( \Gamma^* \) is a Fréchet (complete and metrizable) space. The space \( \Gamma^* \) is equipped with a left shift semigroup \( e \), defined as follows:

\[
(e_t) \tau(\tau) := \gamma(\tau + t) \quad \text{for} \quad \gamma \in \Gamma^*.
\]

(3)

The (bilateral) Laplace transform of a distribution \( s \) is denoted by \( \mathcal{L}(s) \) or by \( \mathcal{S}(s) \). \( C_o \) denotes the closed right-half complex plane

\[
C_o := \{ z \in \mathbb{C} \mid \Re z \geq \sigma \}.
\]

(4)

In particular, \( C_o \) is denoted by \( C \).
As usual, $H^s(C_+)$ denotes the algebra of functions holomorphic and bounded on the open right-half complex plane $\{ z; \Re z > s \}$. When $s = 0$, this space is denoted by $H^0(C_+)$ or simply by $H^0$. For a matrix $A$ over $H^s$, its $H^s$-norm $\| A \|$ is the supremum of the greatest singular value of $A$ on the right-half plane:

$$\| A \| := \sup_{\Re z > s} \sigma_{\text{max}}(A(z)) = \sup_{\Re z > s} \sigma_{\text{max}}(A(z)) \tag{5}.$$ 

The second equality follows from Theorem C, Section 4.8, of [RR].

2. Preliminaries: Pseudorationality

Let us begin our discussion by specifying the class of impulse responses we will work with. We follow the framework given in [Y3], and only outline the ideas. Details can be found in [Y3] [Y4], and [YH1]. An impulse response matrix is a $p \times m$ matrix $A$ with entries being measures on $[0, \infty)$. The singularity of $A$ at the origin, if it exists, is required to be a multiple of the Dirac delta measure $\delta$. The subclass of impulse responses we study in this paper is called pseudorational impulse responses, and it is, roughly speaking, the class of those specified by the convolution equation

$$Q * y = P * u,$$

where $Q$ and $P$ are distributions having bounded support. To be more precise, let us give the following definition.

**Definition 2.1.** Let $A$ be a $p \times m$ impulse response matrix. $A$ is said to be pseudorational if there exist $p \times p$ and $p \times m$ matrices $Q$ and $P$ with entries in $\mathcal{D}(R^2)$ such that

1. $Q^{-1}$ exists over $\mathcal{D}(R)$,
2. $	ext{ord} \det Q = -\text{ord} \det P$,
3. $A$ can be written as

$$A = Q^{-1} * P,$$  \tag{6}

where $\det Q$ and $Q^{-1}$ are taken with respect to convolution. A scalar distribution $q$ satisfying properties 1 and 2 above is said to be of normal type. The pair $(Q, P)$ is said to be approximately left coprime if there exists a sequence of matrices $(R_n)$ and $(S_n)$ over $\mathcal{D}(R^2)$ such that

$$Q * R_n + P * S_n \to \delta; \quad \text{in} \quad \mathcal{D}(R^2).$$

Taking $Q, P$ to be polynomials in the derivative $\mathcal{D}$ of the Dirac delta distribution, we see that finite-dimensional impulse responses are pseudorational. Other typical examples are delay-differential systems. In this case, $Q$ and $P$ are distributions consisting of $\mathcal{D}$ and delay operators, point or distributed. For example, the neutral delay-differential equation

$$x(t) = F_0 x(t) + F_1 x(t - h) + F_{-1} x(t - h) + Go(t) \tag{7}$$
can be expressed in this way by
\[ Q = \delta', I - F_2 \delta - F_1 \delta - F_2 \delta' \quad \text{and} \quad P = G \delta, \]
(8)
More general delay systems and ordinary integrodifferential systems can be handled in this setting [Y1].
To discuss internal exponential stability, we must specify what kind of realization we discuss. Since it is known [F], in general, that a weak notion of canonicity does not leave even the notion of spectrum invariant, this is crucial. Fortunately, we have a rather well-behaved class of realizations naturally associated with pseudorational impulse responses [Y3].
Take a pseudorational \( p \times m \) impulse response matrix \( A \). Consider the following space \( X_A \):
\[ X_A = \{ x \in \Gamma^* | \pi(x) = 0 \}, \]
(9)
where \( \pi \) is the truncation mapping (2), and the closure is taken in \( \Gamma^* \). \( X_A \) is the closure of the space of output functions resulting from the action of past inputs. In particular, if \( A \) is of the form \( A = Q^{-1} \), the space \( X_A \) is denoted by \( X^Q \). Actually, the space \( X^Q \) is given by [Y3]
\[ X^Q = \{ x \in \Gamma^* | \pi(x) = 0 \}. \]
(10)
These spaces are closed subspaces of \( \Gamma^* \), and, furthermore, closed under the left shift semigroup \( \{ \sigma_t \}_{t \geq 0} \) in \( \Gamma^* \). We can then construct a canonical realization \( \Sigma \), by taking \( X_A \) as a state space, and \( \sigma_t \) as the state transition semigroup. This realization is canonical in the sense that (i) it is approximately reachable, i.e., the reachable set is dense in the state space, and (ii) it is topologically observable, i.e., initial states can be recovered from outputs in a well-posed way. If we take \( X^Q \) instead of \( X_A \) \( (X^Q \) always contains \( X_A \) \[ Y3 \]), we obtain another realization, denoted by \( X^Q \), which is still topologically observable but not necessarily approximately reachable. The system equations and detailed constructions can be found in [Y3], and we do not use details of this construction in the following. What concerns us is the fact that \( X_A \) and \( X^Q \) are isomorphic to a Hilbert space [Y3, Proposition 3.5], and hence it is possible to introduce a norm on \( X_A \) and \( X^Q \). Therefore, we can speak of exponential stability of these systems without ambiguity as follows:

**Definition 2.2.** Let \( \Sigma \) be one of the above realizations and let \( \sigma_t \) be the left shift semigroup. We say that \( \Sigma \) is exponentially stable if there exist \( M, \beta > 0 \) such that
\[ |\sigma_t| \leq M e^{-\beta t} \]
(11)
for all \( t \geq 0 \).

3. Stability Theorems

Our target is the following theorem:

**Theorem 3.1.** Let \( A = Q^{-1} \cdot P \) be pseudorational. The canonical realization \( \Sigma_A \) is exponentially stable if one of the following conditions holds:

[Note: The content of the image suggests a continuation or a series of similar conditions, but they are not fully visible and thus not included.]
1. \[
\sup \{ \Re \lambda : \det \tilde{Q}(\lambda) = 0 \} < 0, \quad (12)
\]
2. \[
\frac{1}{\det \tilde{Q}(0)} \in \bigcup_{x_0} H^s(C_x), \quad (13)
\]

These conditions are also necessary if the pair \((\tilde{Q}, \Pi)\) is approximately left coprime in the sense of Definition 2.1.

Clearly, the second condition is stronger than the first. So we need only prove that the first condition is sufficient (for the sufficiency part). Note that \(\bigcup_{x_0} H^s(C_x)\) is strictly contained in \(H^s(C_x)\) (consider \(e^{-i\theta}\) or Logemann's example (1)).

As impulse response \(A\) is pseudorational if and only if each entry is pseudorational [YH2]. Thus we can write \(A = (a_0) = (a_1, \ldots, a_n)\). With this expression, we have the following corollary:

**Corollary 3.2.** Let \(A = (a_0) = (a_1, \ldots, a_n)\) be pseudorational. The canonical realization \(\Sigma_A\) is exponentially stable if one of the following conditions holds for every \(i, j\):

1. \[
\sup \{ \Re \lambda : \tilde{q}_i(\lambda) = 0 \} < 0, \quad (14)
\]
2. \[
\frac{1}{\tilde{q}_i(0)} \in \bigcup_{x_0} H^s(C_x), \quad (15)
\]

These conditions are also necessary if each pair \((a_0, p_j)\) is approximately left coprime in the sense of Definition 2.1.

Corollary 3.2 is a direct consequence of Theorem 3.1 and the following theorem:

**Theorem 3.3 [YH2].** For a pseudorational impulse response, its canonical realization \(\Sigma_A\) is exponentially stable if and only if the canonical realization \(\Sigma_{a_0}\) is exponentially stable for each entry \(a_0\) of \(A\).

Therefore, it suffices to prove Theorem 3.1. Let us first show the necessity part

**Proof of the Necessity of Theorem 3.1.** Suppose that the pair \((\tilde{Q}, \Pi)\) is approximately left coprime, and the canonical realization \(\Sigma_A\) is exponentially stable. Then the canonical realization \(\Sigma_{a_0}\) agrees with \(\Sigma^{\oplus s}\) [Y3, Theorem 4.4], and hence \(\Sigma^{\oplus s}\) must be exponentially stable. Hence it remains stable if we multiply the impulse responses by the vectors \(e^{zt}\) for some \(t > 0\), and the resulting impulse response is still pseudorational (see the proof of Lemma 5.2 below), and the transfer matrix becomes \(\tilde{Q}(s - \beta)\Pi(s - \beta)\). Then, by Theorem 3.9 of [YH1], \(1/\det \tilde{Q}(s - \beta)\) must belong to \(H^s(C_x)\), which means \(1/\det \tilde{Q}(s)\) belongs to \(H^s(C_{s x})\), which completes the proof of necessity.
Now according to Theorem 3.5 in [V4], the canonical realization of \( A = Q^{-1} \cdot P \) is exponentially stable if the same is true of \( q^{-1} \cdot A \), where \( q = \det Q \). Hence if Theorem 3.1 holds for \( A = q^{-1} \), then the canonical realization for \( A = Q^{-1} \cdot P \) is also exponentially stable.

Therefore, it is enough to prove the sufficiency of Theorem 3.1 for the case \( A = q^{-1} \) for a scalar distribution satisfying Definition 2.1. From then on, we assume that \( A \) is of this form.

The rest of the paper is devoted to the proof of Theorem 3.1. To this end, we need some advanced complex analytic properties of transfer functions of pseudorandom input and noise, and this is the theme of the next section.

4. Transfer Functions and Infinite Products

The following Paley-Wiener theorem is the key to our subsequent developments.

**Theorem 4.1** (Paley-Wiener-Schwartz [S2]). A complex function \( \phi(s) \) is the (bilateral) Laplace transform of a distribution \( \psi \) in the space \( \mathcal{E}(\mathbb{R}^+) \) of distributions having compact support contained in \( (-\infty, 0) \) if and only if it is an entire function such that

\[
|\phi(s)| \leq C(1 + |s|)^m e^{\alpha |s|}, \quad Re \ s \geq 0,
\]

\[
\leq C(1 + |s|)^m, \quad Re \ s \leq 0,
\]

for some constants \( C, \alpha > 0 \) and nonnegative integer \( m \). Furthermore, if this estimate is satisfied, then the support of \( \psi \) is contained in \( (-\alpha, 0) \). We call such complex analytic functions as the Paley-Wiener class.

Therefore, for a pseudorandom impulse response \( A = q^{-1} \cdot p \), its transfer function (i.e., Laplace transform) is the ratio of two entire functions \( p(s) \) and \( q(s) \), with exponential growth. In particular, if \( \delta \) is of the form \( A = q^{-1} \), then \( \mathcal{E}(\delta) \) is the reciprocal of such a function.

**Remark 4.2.** In particular, if \( q \) is a measure, i.e., a distribution of order 0, then the integer \( m \) above can be taken to be zero. Since \( supp e \subset [a, b] \) for some \( a, b < \infty \), this follows readily from the definition of the Laplace transform:

\[
|\phi(s)| := \left| \int_{-\infty}^{\infty} e^{-st} dp \right| \leq \int_{-\infty}^{\infty} e^{-st} dt \int |p| dt.
\]

The right-hand side is bounded whenever \( Re \ s \) is bounded from below.

An easy corollary of the Paley-Wiener theorem is the following.

**Corollary 4.3.** Let \( Q \) be a matrix over \( \mathcal{E}(\mathbb{R}^+) \). A function \( x(t) \in \Gamma' \) belongs to the space \( X^\infty \) defined by (10) if and only if the Laplace transform \( \mathcal{L}(Q \cdot x)(s) \) of each entry of \( Q \cdot x \) is in the Paley-Wiener class.
Proof. Without loss of generality, we can assume that $Q$ is a scalar distribution. Since both $Q$ and $x$ have support bounded on the left, their convolution $Q * x$ also has support bounded on the left. By definition, $n(Q * x) = 0$ if and only if $\text{supp}(Q * x) \subset (-\infty, 0]$. Therefore $x \in \mathbb{R}^2$ and only if $Q * x$ has compact support contained in $(-\infty, 0]$, i.e., its Laplace transform $\mathcal{L}(Q * x)(s) \in \mathcal{P}$.

The following lemma claims that the Paley–Wiener class is closed under finitely many cancellations of zeros, and this fact is frequently used in subsequent sections.

**Lemma 4.4.** Let $\varphi(s)$ be in the Paley–Wiener class. Suppose that $\lambda$ is a zero of $\varphi(s)$. Then the function $\varphi(s) := \varphi(s)/(s - \lambda)$ also belongs to the Paley–Wiener class.

**Proof.** Clearly, $\varphi(s)$ is an entire function of $s$. Let $C, a, m$ be the constants in the Paley–Wiener estimate (16) of $\varphi(s)$. Since $\varphi(s)$ is continuous, it is bounded by a constant $M$ on the disc $|s| \leq 1$ of radius 1 centered at $\lambda$. Outside this disc, we have

$$|\varphi(s)| \leq \frac{|\varphi(s)|}{|s - \lambda|} \leq |\varphi(s)|$$

because $|s - \lambda| > 1$. This, along with (16), implies the Paley–Wiener estimate for $\varphi(s)$.

To prove stability, we need an estimate of $\mathcal{L}(\varphi(s))$ on the imaginary axis. What can we say about the growth along the imaginary $s$-axis when $\varphi(s)$ has no poles in $\mathbb{C}$? In general, this is a difficult problem, and even for the restricted class of Laplace transforms, singularities do not govern the growth order. We can give an example (see p. 58 of [W2]) of a Laplace transform that has 0 as the abscissa of convergence $\gamma_0$ and has no singularities on the whole complex plane. This is a typical question when dealing with irrational transfer functions.

Fortunately, we are dealing with a more specialized class of functions, i.e., entire functions of exponential type. This actually leads us to a connection with a classical theory of entire functions of finite order. This class possesses an infinite-product representation which gives a precise characterization of growth order in terms of distribution of zeros. Roughly speaking, $\varphi$ is this property that enables us to give the desired growth estimate on the imaginary axis. Let us start by recalling the following definitions.

**Definition 4.5.** Let $f(s)$ be an entire function. The function $f(s)$ is said to be of order at most $\rho$ if for every $\epsilon > 0$ there exists $r > 0$ such that

$$|f(s)| < e^{\rho |s|^\epsilon} \quad \text{for} \quad |s| > r.$$  

in particular, $f(s)$ is said to be of exponential type if there exists $K, \rho > 0$ such that

$$|f(s)| \leq Ke^{\rho |s|}.$$  

By the Paley–Wiener theorem, the Laplace transform of a distribution $q \in \mathcal{E}'(\mathbb{R}^+)$ is always of exponential type.
Remark 4.6. The above definition of order should not be confused with the order of a distribution. The former gives the estimate of the exponent of a function \( f(z) \), while the latter specifies the number of differentiations contained in \( q \). Hence the order of a distribution \( q \) and the order of its Laplace transform are totally different numbers; for historical reasons, this cannot be avoided. We also note that an entire function of exponential type is of order at most 1, but not conversely.

The following lemma is a special case of the Hadamard factorization where all zeros have negative real parts.

**Lemma 4.7.** Let \( f(z) \) be an entire function of exponential type whose zeros \( \{\lambda_1, \ldots, \lambda_n, \ldots\} \) have negative real parts. Then \( f(z) \) admits the factorization

\[
f(z) = Ce^{\alpha z} \prod_{\lambda_n} \left(1 - \frac{z}{\lambda_n}\right)
\]

or

\[
f(z) = Ce^{\alpha z} \prod_{\lambda_n} \left(1 + \frac{z}{\lambda_n}\right)
\]

where

\[
\sum_{\lambda_n} \frac{|\text{Re } \lambda_n|}{|\lambda_n|^2} < \infty.
\]

**Proof.** By the well-known Hadamard factorization theorem [B, Theorem 2.7.1], \( f(z) \) admits a unique factorization

\[
f(z) = Ce^{\alpha z} \prod_{\lambda_n} \left(1 - \frac{z}{\lambda_n}\right)
\]

or

\[
f(z) = Ce^{\alpha z} \prod_{\lambda_n} \left(1 + \frac{z}{\lambda_n}\right)
\]

depending on whether or not \( \sum_{\lambda_n} |1/\lambda_n| < \infty \).

Moreover, since \( f(z) \) is a function of exponential type, Lindelöf's theorem [B, Theorem 2.10.1] implies that

\[
S(r) = \sum_{\lambda_n} \frac{1}{|\lambda_n|^2} < \infty
\]

is a bounded function in \( r \). Since

\[
\text{Re} \left( \sum_{\lambda_n} \frac{1}{\lambda_n} \right) = \sum \text{Re} \left( \frac{1}{\lambda_n} \right) = \sum \text{Re} \lambda_n / |\lambda_n|^2,
\]

and these terms are all real and negative, the series (22) is absolutely convergent.

The following lemma is the key to the proof of our stability theorem:
Lemma 4.8. Let \( f(s) \) be an entire function of exponential type whose zeros \( \lambda_1, \lambda_2, \ldots \) have negative real parts. Suppose that
\[
0 < C_1 \leq |f(s)| \leq C_2
\]
holds on some line \( L_a = \{ z; \text{Re } s = a \}, a > 0 \), parallel to the imaginary axis. Then
\[
\frac{C_2}{C_1} |\lambda_a - s|^\alpha
\]
is uniformly bounded for \( s \in L_a \).

Proof. Take any \( s_0 \in L_a \), and define
\[
f_{s_0}(s) = \frac{1}{f(s_0)} f(s + s_0).
\]
Then \( f_{s_0}(s) \) and \( 1/f_{s_0}(s) \) are bounded on the imaginary axis, and \( f_{s_0}(0) = 1 \). These bounds are uniform for \( s_0 \in L_a \) by (27). The zeros of \( f_{s_0}(s) \) become \( \lambda_1 - s_0, \lambda_2 - s_0, \ldots \). Since \( f_{s_0}(0) = 1 \) and by (27), the integral
\[
\lim_{s \to \infty} \int_{s_0}^{s} \frac{\log |f_{s_0}(t)| \, dt}{\sqrt{t-s}} = -x^2 I
\]
even if it is finite in the sense of Cauchy’s principal value. Furthermore, \( f_{s_0} \) can be shown to be uniformly bounded for \( s_0 \in L_a \) (see the Appendix for a proof). Let \( n(r) \) be the number of \( \lambda_n - s_0 \) in the disc \( |s| \leq r \). Since (30) exists, the limit
\[
D_n = \lim_{r \to \infty} n(r)
\]
even if it is finite according to Boas [B, Theorem 8.4.1]. Define
\[
C_n = \frac{\sqrt{\pi} \, (\text{Re } \lambda_n - s_0)}{2^{n/2} |\lambda_n - s_0|^\alpha}.
\]
Again, by Theorem 8.4.1 of [B], \( C_n \) exists, and
\[
D_n = 2C_n + 2e^a.
\]
Since \( D_n \) is clearly invariant by changing \( s_0 \) in \( L_a \), and since \( L_a \) is uniformly bounded, \( C_n \) must be uniformly bounded. Since \( \lambda_n - s_0 = \text{Re } \lambda_n - a, \) and \( |\text{Re } \lambda_n - a| \geq |\text{Re } \lambda_1| \), we have the conclusion.

Remark 4.9. In Theorem 8.4.1 of [B] the result is stated for the factorization of type \( (21) \), with exponential factors \( \exp(is/\lambda) \), as the generic case for entire functions of exponential type. The same result can be readily seen to hold for factorizations of type (20) since the factors \( \exp(is/\lambda) \) play no role in the proof given there.

5. Proof of the Main Theorem

We now proceed to the proof of Theorem 3.1. As seen in Section 3, it suffices to show the sufficiency for a scalar pseudorational impulse response of type \( 4 = q^{-1} \).
Before going into details, we present our program since the whole proof is rather lengthy and technical. Suppose we are given such a $q$. Let $\text{ord } q = r$. For the internal exponential stability of the realization $\Sigma(q)$ (which is the canonical realization for $A = q^{-1}$ because $(q, 0)$ is clearly constant), a sufficient condition is that $s' \Phi(s) \in H^\sigma(C, \ell)$. By extracting $\ell$ zeros from $\Phi(s)$, we can rewrite this condition as $1/\ell(s) \in H^\sigma(C, \ell)$, where $\text{ord } \ell(s) = 1$ obtained by dividing $\Phi(s)$ by $(s - \lambda_1) \cdots (s - \lambda_k)$ for suitable zeros $\lambda_1, \ldots, \lambda_k$. As this can always be done, Lemma 5.1, we may assume from the outset that $\text{ord } q = 0$. What we then need to prove is that under the condition of

$$\sup_{\text{Re } \lambda \in C; \ell(s) = 0} < 0$$

$$1/(\ell(s)) \in H^\sigma(C, \ell)$$

$$s' \Phi(s) \in H^\sigma$$

is satisfied if and only if $1/\ell(s) \in H^\sigma$. Since $q_1$ is clearly of order 0, this case reduces to the consideration of $q_1$.

The following lemma shows that the case ord $q > 0$ also reduces to the case ord $q = 0$.

**Lemma 5.1.** Let $q \in \mathcal{E}(R^+)$ be a distribution with ord $q = r > 0$. Then there exist $r$ zeros $\lambda_1, \ldots, \lambda_r$ of $\Phi(s)$ such that $s^{r-1}[\Phi(s)/(s - \lambda_1) \cdots (s - \lambda_r)]$ is of order 0, i.e. a measure.

**Proof.** There exists at least one zero of $\Phi(s)$, otherwise, by Lemma 4.7, $\Phi(s)$ would be a constant multiple of $s^a$ for some $a$ and this $a$ must be real (otherwise, it contradicts the Paley–Wiener estimate (16)). Since $s^a$ is the Laplace transform of $\delta_a$, which is a measure (and of order 0), this contradicts our hypothesis $r > 0$. Repeating the same argument, we see that $\Phi(s)$ has at least $r$ zeros $\lambda_1, \ldots, \lambda_r$. Write $\Phi(s)$ as

$$\Phi(s) = q_1(s) \prod_{j=1}^r (s - \lambda_j).$$

(35)

According to Lemma 4.4, $q_1(s)$ belongs to the Paley–Wiener class, so its inverse Laplace transform $q_1(s)$ belongs to $\mathcal{E}(R^+)$. Since the inverse Laplace transform of $1/(s - \lambda)$ is a differential operator of order $r$, $q_1$ must be of order 0.

The next lemma shows that $1/(\ell(s))$ belongs at least to $H^\sigma(C, \ell)$ for some $\sigma > 0$:
Lemma 5.2. Suppose that \( q \in \mathcal{E}(\mathbb{R}^n) \) is of normal type and that \( \alpha = 0 \). Then there exists \( \sigma > 0 \) such that

\[
\frac{1}{\hat{q}(s)} \in H^{\infty}(C_{\alpha}),
\]

(36)

Proof. Since \( q \) is of normal type, \( \alpha^{-1} = - \alpha = 0 \), so that \( q^{-\frac{1}{\alpha}} \) is also a measure. Since \( X^q \) is isomorphic to a Hilbert space \([Y_3, \text{Proposition } 3.5]\), the transition semigroup \( \sigma_\alpha \) in \( X^q \) has exponential growth, say \( \gamma \). Now take any \( \sigma > \gamma \) and let \( \tau = q^{-\frac{1}{\alpha}}[\hat{q}(s + \sigma)] \). The corresponding impulse response is \( q^{-\frac{1}{\alpha}}[\hat{q}(s + \sigma)] \). We prove that the realization \( \Sigma^\tau \) is exponentially stable. Indeed, take any function \( z(t) \in X^q \). By Corollary 4.3, \( \tau(z) \in X^q \) if and only if \( \tau(z)(s)(t) = \hat{q}(s + \sigma)z(t) \) is an entire function satisfying the Paley-Wiener estimate (16). Put \( z(t) = e^{\tau\alpha}(t)z(t) \). Then \( \tau(z)(s) = \hat{q}(s + \sigma)z(t) \), so that \( \hat{q}(s + \sigma)\hat{z}(t) \) is an entire function satisfying (16). Since \( \hat{q}(s + \sigma) \) is invariant under the change \( s \rightarrow s - \sigma \), \( \hat{z}(t) \) is also an entire function satisfying (16). Thus \( x(t) \in X^q \), and hence \( \hat{z}(t) = e^{\tau\alpha}(t)z(t) \) must belong to \( L^2(0, \infty) \) because the growth rate of \( \hat{z}(t) \) is slower than \( \sigma \). This implies \( X^q \subseteq L^2(0, \infty) \). Now according to Theorem 3.5 in [YH1], this means exponential stability of \( \Sigma^\tau \). Therefore, by Proposition 3.6 in [YH1],

\[
\frac{1}{\hat{q}(s + \sigma)} \in H^{\infty}(C_{\alpha}),
\]

which also yields \( 1/\hat{q}(s) \in H^{\infty}(C_{\alpha}) \), as desired.

Now let \( q \) be a normal type distribution with \( \alpha = r \) that satisfies condition (34), and let \( \alpha \) be the measure given by Lemma 5.1. Since \( s' \) and \( (s - \lambda_1) \cdots (s - \lambda_r) \) are of the same order on \( C_{\alpha} \) (note \( \lambda_\alpha < 0 \) by (34)), \( s'\hat{q}(s) \) and \( 1/\hat{q}(s) \) are of the same growth order on \( C_{\alpha} \). Therefore, our proof of Theorem 3.1 is reduced to showing the following proposition.

Proposition 5.3. Let \( q \in \mathcal{E}(\mathbb{R}^n) \) be a normal type distribution of order \( \alpha \) such that

\[
\sup \{ \Re \lambda \in C; \hat{q}(\lambda) = 0 \} = -c < 0.
\]

(37)

Then

\[
\frac{1}{\hat{q}(s)} \in H^{\infty}(C_{\alpha}),
\]

(38)

Let us first motivate the idea of the proof of Proposition 5.3. Since \( \hat{q}(s) \) is an entire function of exponential type by Theorem 4.1, Lemma 4.7 gives the following infinite dimensional representation for \( \hat{q}(s) \):

\[
\frac{1}{\hat{q}(s)} = \frac{1}{C \omega^\alpha} \prod_{\lambda \neq 0} (1 - \omega \lambda)^{-\alpha} \exp(s/\lambda^\alpha)
\]

(39)

where

\[
\frac{\Re \lambda_\alpha}{|\lambda|^\alpha} < \infty.
\]

(40)
We attempt to estimate the ratio

$$\sup_{\omega = a + jw} \frac{|\hat{f}(\sigma + j\omega)|}{|\hat{f}(\sigma - j\omega)|}$$  \hspace{1cm} (41)$$

Since \(1/\phi(\sigma + j\omega)\) is already known to be bounded by Lemma 5.2, it suffices to show that (41) is bounded.

**Proof of Proposition 5.3.** Write \(s = \sigma + j\omega\). The fraction (41) becomes

$$\frac{1}{\phi(s - \sigma)} \frac{1}{\phi(s)} = e^{\sigma \omega} \frac{1}{\omega} \sqrt{\frac{1 - s/\lambda_s}{1 - (s - \sigma)/\lambda_s}} \exp\left(\frac{s}{\lambda_s}\right)$$  \hspace{1cm} (42)$$

(If \(\phi(\cdot)\) has the factorization of type (28), we do not need the exponential factors \(\exp(s/\lambda_s)\). In view of Remark 4.9, this presents no difficulty.) We need to estimate the growth of the infinite product above on the line \(L_s = \{s; \Re s = \sigma\}\). Denote this product by \(F(s)\). Then

$$|F(s)| = \exp(\log|F(s)|) = \exp(\Re \log F(s))$$

$$= \exp\left(\sum_{\lambda_s} \Re \frac{s}{\lambda_s - s} - \sum_{\lambda_s} \Re \left(\log \left(1 - \frac{s}{\lambda_s}\right) - \log \left(1 - \frac{s}{\lambda_s - s}\right)\right)\right)$$

$$= \exp\left(\sum_{\lambda_s} \frac{s}{\lambda_s} - s - \sum_{\lambda_s} \Re \left(\log \left(1 + \frac{s}{\lambda_s - s}\right)\right)\right)$$  \hspace{1cm} (43)$$

where we take the principal branch for \(\log\). By (40), the first sum on the right-hand side is finite. So we need only estimate the second sum involving \(\log\). By (37) we have

$$\rho := \sup_{\omega \in \mathbb{R}, \sigma = \sigma_0} \left|\frac{s}{\lambda_s - s}\right| < \frac{\sigma}{\sigma + \rho} < 1.$$  \hspace{1cm} (44)$$

Hence the expansion

$$\log \left(1 + \frac{s}{\lambda_s - s}\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{s^n}{\lambda_s^2}$$  \hspace{1cm} (45)$$

is valid. Therefore, we have

$$|F(s)| = \exp\left(\sum_{\lambda_s} \Re \frac{s}{\lambda_s - s} - \sum_{\lambda_s} \Re \left(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{s^n}{\lambda_s^2}\right)\right)$$  \hspace{1cm} (46)$$

It suffices to prove that the latter double infinite sum (46) converges uniformly on \(L_s\) and is bounded there. Recall that \(\hat{s}(\sigma)\) is bounded on \(L_s\) by Remark 4.2, and \(1/\phi(\cdot)\) is also bounded on \(L_s\) by Lemma 5.2. Thus \(\hat{s}(\cdot)\) satisfies the hypotheses of Lemma 4.8. This implies (see also (32)) that the first term

$$\sum_{\lambda_s} \Re \frac{s}{\lambda_s - s} = \frac{\sigma}{\lambda_s} \left(\sigma - \sigma_0\right)$$

converges absolutely and uniformly bounded on \(L_s\). Since \(\Re \lambda_{s} - \sigma \geq c + \sigma > \sigma\),

$$\sum_{\lambda_s} \frac{1}{\lambda_s - s}$$  \hspace{1cm} (47)$$
is also uniformly bounded. By (44), we have
\[
\begin{array}{ll}
\left| \frac{\sigma}{\lambda_k - s} \right|^2 & \leq \frac{\rho^{k-2}}{\lambda_k - s} \left| \frac{\sigma}{\lambda_k - s} \right|^2 \quad (48)
\end{array}
\]
for all \( k \geq 2 \). Hence
\[
\frac{1}{\lambda_k} \frac{\sigma}{\lambda_k - s} \leq \frac{1}{\lambda_k} \frac{\rho^{k-1}}{\lambda_k - s} \left| \frac{\sigma}{\lambda_k - s} \right|^2, \quad (49)
\]
where the right-hand side converges absolutely and uniformly on \( L_\sigma \). Therefore, the double infinite sum (46) converges absolutely and uniformly on \( L_\sigma \) and is bounded there. Thus \( 1/\sigma(s) \) is bounded on the imaginary axis, belongs to \( H^\infty(C_\sigma) \), and is analytic in \( C_\sigma \). Then an easy application of the Phragmén–Lindelöf theorem [8] implies that \( 1/\sigma(s) \) is bounded on \( [\delta; \infty, \Re s \leq \sigma] \), so indeed belongs to \( H^\infty(C_\sigma) \). This completes the proof of Proposition 5.3, and hence that of Theorem 3.1.

6. Examples

Example 6.1. Let us examine Logemann’s example (5). \( W(s) = \frac{1}{s + 1} \frac{1}{s(1 - e^{-s}) + 1} \). (50)

from our viewpoint. We have set \( \sigma = a = 1 \) for brevity. Rewrite (50) as

\[
W(s) = \frac{1}{s + 1} \frac{e^s}{s(e^s - 1) + e^s}. \quad (51)
\]

Taking the inverse Laplace transform, we have

\[
\mathcal{L}^{-1}(W) = (s + 1)^{-1} \delta(s) + \delta(s - 1) - \delta(s - 2). \quad (52)
\]

(Recall \( \mathcal{L}[\delta(s)] = e^s \). This is clearly pseudorandom.) Set

\[
q_1 := s + 1, \quad q_2 := s(s - 1) - \delta(s - 2), \quad p := \delta(s - 1), \quad q := q_1 + q_2.
\]

Then ord \( q_1 = 1 \), ord \( q_2 = 1 \), and ord \( q = 2 \). Clearly, there is no common zero between \( q(s) \) and \( p(s) \), and, furthermore, the least upper bound of the support of \( q \) is zero. This means, according to Theorem 4.8 and Corollary 4.9 of [Y4], that the pair \((q, p)\) is approximately coprime, and the realization \( \Sigma \) is canonical. Therefore, the stability of the canonical realization can be checked by the zeros of \( q(s) \).

As Logemann [L2] has shown, all poles of \( W(s) \) are stable, but there exists a sequence of poles approaching the imaginary axis. Therefore, the system cannot be exponentially stable by Theorem 3.1. How is this reflected upon the \( H^\infty \)-property of the transfer function? As shown in [L2], \( W(s) \), and hence \( 1/\sigma(s) \), is in \( H^\infty(C_{\sigma}) \). In view of Theorem 3.1 or Theorem 3.12 in [Y3], this is not enough for stability. Since ord \( q = 2 \), we
must have

\[ \frac{x^2}{q(s)} \in H^s(C_1) \quad \text{or} \quad \frac{1}{q(s)} \in H^s(C_1) \] (53)

for exponential stability, where \( q(s) \) is the function obtained by extracting two (stable) zeros from \( q(s) \). In fact, as in [L2], it is possible to show

\[ s \cdot q(s) \in H^s(C_1) \]

Thus the canonical realization is not exponentially stable. Actually, Legemann's example is made is such a way that

\[ \frac{1}{q(s)} \in H^s(C_1) \] (54)

but

\[ \frac{1}{s + 1} \cdot q(s) \in H^s(C_1) \] (55)

so that the overall transfer function belongs to \( H^s(C_1) \). Multiplying the stable factor \( s + 1 \) can thus make a non-\( H^s \) transfer function \( H^s \), but this does not mean, of course, that we can embed an unstable system into a stable system just by multiplying \( s + 1 \) to the denominator. If we multiply \( q(s) \) by \( s + 1 \), then one order higher condition \( s(s + 1)q(s) \in H^s \) becomes necessary for stability, which is equivalent to \( 1/q(s) \in H^s \).

Let us also examine this example from the state-space point of view. It is known that a system \( (F, G, H) \) must be exponentially stable if the resolvent operator \((I - F)^{-1}\) belongs to \( H^s(C_1) \) (in the sense of operator norm) (see, e.g., [P] and [W1]). This appears to be somewhat mysterious against the property \( R(s) = H^s(C_1) \), because in this case there is no pole-zero cancellation in \( H(I - F)^{-1} \) or \((I - F)^{-1}G \) so it deserves more detailed analysis.

The transfer function \( W(s) \) admits the following delay-differential equation description [L2]:

\[
\begin{align*}
\dot{x}_1(t) &= -x_1(t) + w(t), \\
x_2(t) &= x_1(t) - x_2(t) + x_2(t - 1), \\
y(t) &= x_2(t).
\end{align*}
\] (56)

The canonical realization \( \Sigma^* \) can be computed following [YU], and it becomes the following system:

\[
\begin{align*}
\begin{bmatrix}
\dot{w}_1 \\
\dot{w}_2 \\
\dot{x}_2(t)
\end{bmatrix} &= \begin{bmatrix}
w_1 \\
w_2 \\
(I - z)^{-1}(-1) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\end{bmatrix} w(t), \\
y &= w_2 + x_2(t).
\end{align*}
\] (57)
where

\[(w_1, w_2, z) \in X \ni R \times R \times L^2[-1, 0]\]  

and the domain \(D(F)\) of the right-hand side operator \(F\) in (57) is given by

\[D(F) = \{(w_1, w_2, z) \in X; z(1) \in W^2_0[-1, 0]\}\]  

and \(z(0) = w_2 + z(-1)\). (60)

Here \(W^2_0[-1, 0]\) denotes the first-order Sobolev space. The variables \(x_1, x_2\) and \(w_1, w_2\) are related by \(x_1 = w_1\) and \(x_2 = w_2 + z(-1)\).

Solving, for example,

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}^{1/2}
= 
\begin{bmatrix}
w_1 \\
w_2
\end{bmatrix}
\begin{bmatrix}
0 \\
z(1)
\end{bmatrix}
\]

for \((w_1, w_2, z) \in D(F)\), we obtain

\[
w_1 = 0, \\
w_2 = -e^{-s} \frac{1}{s(1 - e^{-s}) + 1}, \\
z(0) = e^{s}\frac{1}{s(1 - e^{-s}) + 1}.
\]

In view of (54), the latter two quantities do not belong to \(H^2(C, \mathcal{L})\). This shows that the resolvent operator \((\omega - F)^{-1}\) is not in \(H^2(C, \mathcal{L})\). Here again, the factor \(1/(s + 1)\) disappears in the resolvent operator. Thus the growth (on the imaginary axis) of the resolvent operator cannot be altered by putting in this extra factor, even though it affects the growth of the transfer function.

**Example 6.2.** Let us now give another example where condition (12) or (13) guarantees exponential stability. Instead of (51), consider

\[W(s) = \frac{1}{s + 1} \left( e^s + e^{-s} \right) e^s \]

where \(0 < s < 1\). This amounts to changing (57) to

\[
\begin{bmatrix}
w_1 \\
z(1)
\end{bmatrix} = 
\begin{bmatrix}
-\frac{1}{w_1} \\
0
\end{bmatrix} 
+ 
\begin{bmatrix}
1 \\
0
\end{bmatrix} \begin{bmatrix}
0 \\
w(1)
\end{bmatrix}.
\]

To guarantee the desired stability, it suffices to show

\[s(e^s - s) + e^s \in H^\gamma(C, \mathcal{L})\]

for some \(\gamma > 0\). Take any \(\gamma < \min\left\{ \frac{1}{2}, -\log s \right\}\) (note \(s < 1\)). For any \(s = -\delta + jo\),
0 < \delta \leq \gamma, we have
\begin{align*}
|s + \delta| \geq |s + i\omega| &= \sqrt{s^2 + (\delta + i\omega)^2} \\
&= \sqrt{s^2 + \delta^2 + 2\delta i\omega} \\
&\geq \sqrt{1 + \delta^2 / \omega^2} \\
&\geq |e^{s} - a| \geq |e^{s} - a|,
\end{align*}
(66)

because \delta < 1. Since \gamma < -\log a, the last term is positive. This yields
\begin{align*}
\sup_{-\gamma \leq \nu \leq \nu} \left| \frac{1}{s(e^{s} - a) + e^{\nu}} \right| &\leq \sup_{-\gamma \leq \nu \leq \nu} \left| \frac{1}{s + 1} \right| \left| \frac{1}{e^{s} - (s + 1)a} \right| \\
&\leq \frac{1}{1 - \gamma} \left| e^{s} - a \right|
\end{align*}
so this (65) is proved. Therefore, condition (13), and hence (12), is satisfied, and system (64) is exponentially stable.

7. Concluding Remarks

We have shown that exponential stability of the integral canonical realization \( \Sigma_4 \) of a pseudorational impulse response \( Q^{-1} * P \) can be checked either by the location of the spectrum, or by the \( H^\infty \)-property
\[
\frac{1}{\det \Omega(s)} \in \bigcup_{s \in \gamma} \text{H}^\infty(\mathbb{C}_+)\n\]
These conditions are also necessary if \( Q \) and \( P \) are approximately left coprime. According to a result in [Y4], this is essentially the question of pole-zero cancellation between \( Q(s) \) and \( P(s) \). Since this class can contain infinitely many unstable zeros, this causes a severe technical difficulty. It is an interesting open problem to give a criterion which is independent of such pole-zero cancellation behavior. We refer the reader to [Y5] and [YH2] for an effort in this direction.

Acknowledgments. The first version of the proof of Proposition 5.3 contained an error; therefore, the proof given in the preliminary version [Y6] must be modified accordingly. The author wishes to thank the referee who pointed out the flaw in the proof. A preliminary version of this paper [Y6] was presented at MTNS-89, held June 19-23, 1989, at Amsterdam. The author wishes to thank the Symposium committee for their support, and also thanks the Japan Ministry of Education for the travel grant.

Appendix. Uniform Boundedness of Integral (30)

In view of condition (27), integral (30) is uniformly bounded for \( R \to \infty \). So it is
enough to show that the principal value

\[ \text{p.v.} \int_{-\infty}^{\infty} \frac{\log |f_{\alpha}(\xi)|}{\alpha^2} \, d\alpha \]

is finite and uniformly bounded in \( s_0 \in L_0 \) for some (uniformly) small enough \( \varepsilon \). Since \( \log |f_{\alpha}(\xi)| = \text{Re}(\log f_{\alpha}(s)) \), it suffices to prove the claim for \( \log f_{\alpha}(\xi) \) in place of \( \log |f_{\alpha}(\xi)| \).

Write

\[ f_{\alpha}(s) = 1 + s g_{\alpha}(s), \quad g_{\alpha}(s) := \frac{1}{f(\alpha s)} \int f(s + s_0) - f(s) \, ds \]

The function \( g_{\alpha}(s) \) is clearly an entire function of exponential type. In view of the mean value theorem, \( g_{\alpha}(\xi) \) is uniformly bounded for \(|\alpha| \leq 1 \) and \( s_0 \in L_0 \) if the derivative \( f' \) is bounded on \( L_0 \). But this follows directly from Bernstein's theorem [B, Theorem 1.1.2]. Hence

\[ |g_{\alpha}(\xi)| \leq M \quad \text{for all} \quad |\alpha| \leq 1, \quad s_0 \in L_0 \quad (67) \]

for some \( M > 0 \). Now take any \( \varepsilon < \min \{1/M, 1\} \). Then we have

\[ \log f_{\alpha}(s) = s g_{\alpha}(s) + s^2 (g_{\alpha}(s))^2 \sum_{n \in \mathbb{Z}} \left( \frac{-1}{2\pi} \right)^n (s g_{\alpha}(s))^n = s g_{\alpha}(s) + s^2 (g_{\alpha}(s))^2 \psi_{\alpha}(s) \]

for \( s = j\alpha, |\alpha| \leq \varepsilon \). Now set

\[ \phi_{\alpha}(s) = g_{\alpha}(s) + s h_{\alpha}(s). \]

As before, we have

\[ |h_{\alpha}(\xi)| \leq K, \quad |\alpha| \leq \varepsilon, \quad s_0 \in L_0 \quad (68) \]

for some \( K > 0 \), and \( \phi_{\alpha}(\xi) \) is clearly uniformly bounded for \(|\alpha| \leq \varepsilon \) and \( s_0 \in L_0 \) because \( |h_{\alpha}(\xi)| < 1 \) \( \forall \alpha \in \mathbb{C} \). Since

\[ \text{p.v.} \int_{-\infty}^{\infty} h_{\alpha}(\xi) \, d\alpha = 0, \]

we have

\[ \text{p.v.} \int_{-\infty}^{\infty} \log f_{\alpha}(\xi) \, d\alpha = - \int_{-\infty}^{\infty} h_{\alpha}(\xi) \, d\alpha - \int_{-\infty}^{\infty} (g_{\alpha}(\xi))^2 \psi_{\alpha}(\xi) \, d\alpha. \]

By (67), (68), and the uniform boundedness of \( \phi_{\alpha}(s) \), the right-hand side is uniformly bounded in \( s_0 \in L_0 \), and this completes the proof.

References


Equivalence of Internal and External Stability for a Class of Distributed Systems


